

Rateless Coding for MIMO Block Fading Channels

Yijia Fan*, Lifeng Lai*, Elza Erkip*[†], H. Vincent Poor*

*Department of Electrical Engineering, Princeton University, Princeton, NJ, 08544, USA

Email: {yijiafan,llai,poor}@princeton.edu

[†]Department of Electrical and Computer Engineering, Polytechnic University, Brooklyn, NY, 11201, USA

Email: elza@poly.edu

Abstract—In this paper the performance limits and design principles of rateless codes over fading channels are studied. The diversity-multiplexing tradeoff (DMT) is used to analyze the system performance for all possible transmission rates. It is revealed from the analysis that the design of such rateless codes follows the design principle of approximately universal codes for parallel multiple-input multiple-output (MIMO) channels, in which each sub-channel is a MIMO channel. More specifically, it is shown that for a single-input single-output (SISO) channel, the previously developed permutation codes of *unit* length for parallel channels having rate LR can be transformed *directly* into rateless codes of length L having multiple rate levels $(R, 2R, \dots, LR)$, to achieve the DMT performance limit.

I. INTRODUCTION

A. Background

Rateless codes present a class of codes that can be truncated to a finite number of lengths, each of which has a certain likelihood of being decoded to recover the entire message. Compared with conventional coding schemes having a single rate R , such codes can achieve multiple rate levels $(R, 2R, \dots, LR)$, depending on different channel conditions. A rateless code is said to be *perfect* if each part of its codeword is capacity achieving. Compared with conventional codes, rateless codes offer a potentially *higher rate*. Several results have been obtained on the design of perfect rateless codes over erasure channels and additive white Gaussian noise (AWGN) channels (see [6] and the references therein).

Unlike in the fixed channel scenario, non-zero error probability always exists in fading channels, when the instantaneous channel state information (CSI) is not available at the transmitter and a codeword spans only one or a small number of fading blocks. In this scenario, it is well known that there is a fundamental tradeoff between the information rate and error probability over fading channels, which can be characterized as the diversity-multiplexing tradeoff (DMT) [1].

Definition 1 (DMT): Consider a multiple-input multiple-output (MIMO) system and a family of codes C_η operating at average SNR η per receive antenna and having rates R . The multiplexing gain and diversity order are defined as

$$r \triangleq \lim_{\eta \rightarrow \infty} \frac{R}{\log_2 \eta} \quad \text{and} \quad d \triangleq - \lim_{\eta \rightarrow \infty} \frac{\log_2 P_e(R)}{\log_2 \eta}, \quad (1)$$

where $P_e(R)$ is the average error probability at the transmission rate R .

The DMT is an effective performance measure for implementing the rateless coding principles in a fading channel. Two

main concerns naturally arise: (a) determining the DMT limit for rateless coding with finite numbers of blocks in a fading environment and discovering how it performs with regard to conventional schemes; and (b) determining DMT achieving codes that are simple (in the sense of encoding and decoding complexity).

B. Contributions of the Paper

In this paper, we analyze the DMT performance of rateless codes. The results show that, compared with conventional coding schemes having multiplexing gain r_n , rateless codes having multiple rates $(r_n, 2r_n, \dots, Lr_n)$ offer an *effective* multiplexing gain r of Lr_n , given the same diversity gain at every rate, when r_n is *small*. As r_n increases, the performance of rateless codes degrades and ultimately becomes the same as that of conventional schemes. Also while increasing L lifts up the overall system DMT curve, it does not necessarily improve the system multiplexing gain for every fixed value of r_n . It is then revealed that the design of such rateless codes follows the principle of parallel channel codes that are *approximately universal* [3] over fading channels. More specifically, it is shown that for a single-input single-output (SISO) channel, the formerly developed *unit* length permutation codes for parallel channels [3] having rate LR can be transformed *directly* into rateless codes of L -length having multiple rate levels $(R, 2R, \dots, LR)$, to achieve the DMT performance limit. For multiple-input multiple-output (MIMO) channels, the results in the paper suggest a type of rateless codes that may be viewed as a combination of conventional MIMO space-time codes and parallel channel codes, both of which have been designed for fading channels.

C. Related Work

The performance of rateless coding over fading channels has also been considered in [4], in which the throughput and error probability are discussed. However, the tradeoff between these two was not analyzed explicitly. For example, the results in [4] shows that increasing the value of L will decrease the system error probability in certain scenario and is therefore desirable. In this paper we show that while this discovery is true, the system throughput, i.e., multiplexing gain might decrease when L becomes larger for every fixed value of r_n . Overall, our results reveal that the optimal design of rateless codes requires the consideration of both r_n and L .

Rateless coding may be considered as a type of Hybrid-ARQ scheme [2]. The DMT for ARQ has been revealed in [2]. However, it will be shown in the paper that this DMT curve was incomplete and represents the performance only when $r_n < \min(M, N)/L$ in which M and N are the number of transmit and receive antennas. The *complete* DMT curve for rateless coding including those parts for higher r_n has never been revealed before, and will be shown in this paper. In addition to this, the results in this paper also offer a relationship between the design parameter (i.e., r_n and L) and the effective multiplexing gain r of the system, thus offer further insights into system design and operational meaning compared to conventional coding schemes. Furthermore, we suggest new design solutions for rateless codes. Previous work on finite-rate feedback MIMO channels relies on either power control or adaptive modulation and coding (e.g., [5]), which are not necessary for our scheme.

The rest of this paper is organized as follows. The system model is proposed in Section II. In Section III, the DMT performance of rateless codes is studied. In Section IV, design of specific rateless codes over fading channels is discussed. Finally, concluding remarks are made in Section V.

II. SYSTEM MODEL

We consider a frequency-flat fading channel with M transmit antennas and N receive antennas. We assume that the transmitter does not know the instantaneous CSI on its corresponding forward channels, while CSI is available at the receiver. Each message is encoded into a codeword of L blocks. Each block takes T channel uses. We assume that the channel remains static for the entire codeword length (i.e., L blocks)¹. The system input-output relationship can be expressed as

$$\mathbf{Y} = \sqrt{\frac{P}{M}} \mathbf{H} \mathbf{X} + \mathbf{N} \quad (2)$$

where $\mathbf{X} \in \mathbb{C}^{M \times TL}$ is the input signal matrix; $\mathbf{H} \in \mathbb{C}^{N \times M}$ is the channel transfer matrix whose elements are independent and identically distributed (i.i.d.) complex Gaussian random variables with zero means and unit variances; $\mathbf{N} \in \mathbb{C}^{N \times TL}$ is the AWGN matrix with zero mean and covariance matrix \mathbf{I} ; and $\mathbf{Y} \in \mathbb{C}^{N \times TL}$ is the output signal matrix. P is the total transmit power, which also corresponds to the average SNR η (per receive antenna) at the receiver.

The input signal matrix \mathbf{X} can be written as

$$\mathbf{X} = [\mathbf{X}_1 \quad \cdots \quad \mathbf{X}_L] \quad (3)$$

where $\mathbf{X}_l \in \mathbb{C}^{M \times T}$ is the codeword matrix being sent during the l th block, and its corresponding receiver noise matrix is denoted by $\mathbf{N}_l \in \mathbb{C}^{N \times T}$. We impose a power constraint on

each \mathbf{X}_l so that²

$$E \left[\frac{1}{T} \|\mathbf{X}_l\|_F^2 \right] \leq M, \quad (4)$$

for $l = 1, \dots, L$.

A. Conventional Schemes

Assume that the transmitter sends the codeword at a rate R bits per channel use. A message of size RT is encoded into a codeword \mathbf{X}_l ($l = 1, \dots, L$) and transmitted in T channel uses. An alternative method is to encode a message of size RLT into \mathbf{X} . Both encoding methods will offer the same performance provided that T is sufficiently large.

B. Rateless Coding

When rateless coding is applied, we wish to decode a message of size RLT with the codeword structure as shown in (3). During the transmission, the receiver measures the total mutual information I between the transmitter and the receiver and compares it with RLT after it receives each codeword block \mathbf{X}_l . If $I < RLT$ after the l th block, the receiver remains silent and waits for the next block. If $I \geq RLT$ after the l th block, it decodes the received codeword $[\mathbf{X}_1 \quad \cdots \quad \mathbf{X}_l]$ and sends one bit of positive feedback to the transmitter. Upon receiving the feedback, the transmitter stops transmitting the remaining part of the current codeword and starts transmitting the next message immediately.

Unlike conventional schemes, this process will bring multiple rate levels $(R, 2R, \dots, LR)$. For example, if $I \geq RLT$ after the first block is received (i.e., $l = 1$), the receiver will be able to decode the entire message and the rate becomes LR . Similar observations can be made for $l = 2 \dots L$. Therefore, compared with conventional schemes, the corresponding transmission rate achieved by using rateless codes is always *equal or higher*. Specifically, we define the multiplexing gain for each rate level as $(r_n, 2r_n, \dots, Lr_n)$ where

$$r_n \triangleq \lim_{\eta \rightarrow \infty} \frac{R}{\log_2 \eta}.$$

Later we will show through the DMT analysis that rateless coding can retain the same diversity gain as conventional schemes, but with a much higher multiplexing gain especially when the corresponding r_n is low.

III. PERFORMANCE ANALYSIS

Denote by ε_l the decoding error when decoding is performed after the l th block ($0 \leq l \leq L$) and by $\Pr(\varepsilon_l, l)$ the joint probability that a decoding error occurs and decoding is achieved after l th block. The system overall error probability can be expressed as

$$P_e = \sum_{l=1}^L \Pr(\varepsilon_l, l).$$

¹Note, however, that the analysis in the paper can be extended straightforwardly to a faster fading scenario in which the channel varies from block to block during each codeword transmission.

²Note that this is a more strict constraint than letting $E \left[\frac{1}{TL} \|\mathbf{X}\|_F^2 \right] \leq M$, which offers at least the same performance.

Define $p(l)$ ($0 \leq l \leq L$) to be the probability with which $I < RLT$ after the l th block, and note that $p(0) = 1$. Following the steps in Section II.B in [2], the average transmission rate for *each message* in bits per channel use is given by

$$\bar{R} = \frac{RL}{\sum_{l=0}^{L-1} p(l)}. \quad (5)$$

Note that this \bar{R} describes the average rate with which the message is removed from the *transmitter*; i.e., it quantifies how quickly the message is decoded at the receiver. We define the effective multiplexing gain of the system as

$$r = \lim_{\eta \rightarrow +\infty} \frac{\bar{R}}{\log_2 \eta}.$$

Define $f(k)$ to be the piecewise linear function connecting the points $(k, (M-k)(N-k))$ for integral $k = 0, \dots, \min(M, N)$. Recall that a conventional scheme operating at multiplexing gain r_n ($0 \leq r_n \leq \min(M, N)$) would have the diversity gain $f(r_n)$. The following theorem shows the performance of rateless coding for $0 \leq r_n < +\infty$.

Theorem 1: Assume a sufficiently large T . For rateless codes having multiple multiplexing gain levels $(r_n, 2r_n, \dots, Lr_n)$, the corresponding DMT can be expressed as (r, d) where

$$r = r_n \cdot \frac{L}{l} \quad \text{and} \quad d = f\left(\frac{lr}{L}\right)$$

for

$$\frac{l-1}{L} \min(M, N) \leq r_n < \frac{l}{L} \min(M, N)$$

and $l = 1, 2, \dots, L$. Finally, $d = 0$ for $r_n \geq \min(M, N)$.

Proof: See Appendix A. ■

Note that for rateless coding to achieve the performance in *Theorem 1*, we do not necessarily require $T \rightarrow +\infty$. As long as T is large enough such that the error probability $\Pr(\varepsilon_l, l) \leq \eta^{f(r_n)}$ for each l , the DMT in *Theorem 1* can be achieved. While the minimal T for a general MIMO channel when applying rateless coding is unknown to the authors, it will be shown later that for SISO channels, $T = 1$ is sufficient to achieve the optimal DMT in *Theorem 1*.

Comparing rateless coding with conventional schemes, it can be shown that for $0 \leq r_n < \min(M, N)/L$, $r = Lr_n$ for $d = f(r_n)$. In this scenario rateless coding can improve the multiplexing gain up to L times that of conventional schemes, given the same diversity gain. Fig. 1 gives an example when $M = N = 2$ and $L = 2$, and $0 \leq r_n \leq 1$. The operating point A in the curve for a conventional scheme for $0 \leq r_n \leq 1$ corresponds to point B in the curve for rateless coding.

An important observation from *Theorem 1* is that the system performance will not be improved after r_n (almost) reaches $\min(M, N)/L$, as the optimal DMT is already achieved by using rateless coding. This is mainly due to the fact that the first block can no longer support the message size when the message rate reaches $\min(M, N)/L$. Thus the system

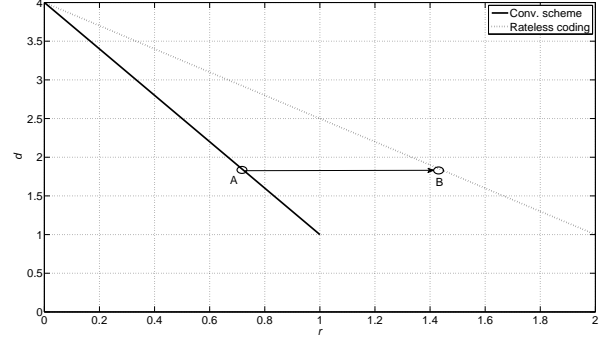


Fig. 1. The DMTs for conventional schemes and rateless coding for $0 \leq r_n \leq 1$. $M = N = 2$, $L = 2$.

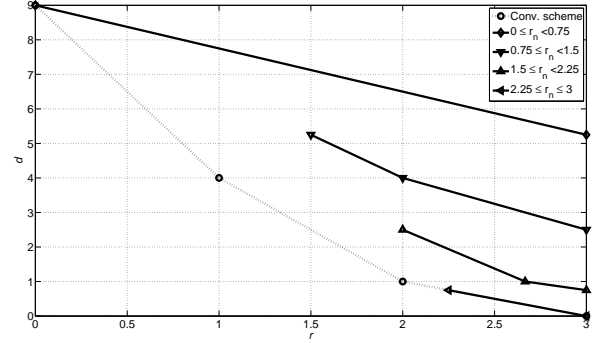


Fig. 2. The DMTs for different schemes for $0 \leq r_n \leq 3$. $M = N = 3$, $L = 4$.

multiplexing gain *decreases* for the same diversity gain, and finally offers the same DMT as conventional schemes when the first $L - 1$ blocks all fail to decode the message. Fig. 2 shows an example when $M = N = 3$, $L = 4$. This observation also implies that for any fixed value of r_n , simply increasing the value of L does *not necessarily* improve the system DMT performance. Although the overall system DMT will increase when L is larger, the multiplexing gain might decrease for certain fixed values of r_n . A convenient choice for L would be in the region of $L < \min(M, N)/r_n$. However, note that the maximal multiplexing gain $\min(M, N)$ can be achieved only with zero diversity gain, and this happens when $r_n = \min(M, N)$ *regardless of* the value of L .

IV. DESIGN OF RATELESS CODES

Note that codewords \mathbf{X}_i ($1 \leq i \leq L$) in (3) are transmitted through different channels that are *orthogonal* in time. This is analogous to transmitting \mathbf{X}_i through different channels that are parallel in *space*. In the (space) parallel channel model, elements in $\{\mathbf{X}_i\}$ can be jointly (simultaneously) decoded. However, for the channel model considered in this paper, which we now call the *rateless channel*, the decoding process needs to follow certain direction in time, i.e., we start decoding from \mathbf{X}_1 , then $[\mathbf{X}_1 \ \mathbf{X}_2]$ if \mathbf{X}_1 is not decoded, etc. This comparison implies that while good parallel channel codes

can be used as the basis for rateless coding, they might need modifications in order to offer good performance over the rateless channel.

Specifically, for the rateless channel expressed in the form of (2), we consider the corresponding parallel MIMO channel, in which each sub-channel is a MIMO channel, having the following input-output relationship:

$$\mathbf{Y} = \sqrt{\frac{P}{M}} \begin{pmatrix} \mathbf{H} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{H} \end{pmatrix} \begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_L \end{pmatrix} + \begin{pmatrix} \mathbf{N}_1 \\ \vdots \\ \mathbf{N}_L \end{pmatrix} \quad (6)$$

where \mathbf{H} , \mathbf{X}_i and \mathbf{N}_i are the same as those in (2). It is easy to see that the DMT for this system is $d = f\left(\frac{r}{L}\right)$ for $0 \leq r \leq L \min(M, N)$. Assuming a code that achieves this DMT, when we implement its transformation $[\mathbf{X}_1 \cdots \mathbf{X}_L]$ into the rateless channel having multiple rates $(r_n, 2r_n, \dots, Lr_n)$, it is not difficult to show that

$$\Pr(\varepsilon_L, L) \leq \eta^{-f(r_n)}. \quad (7)$$

In order to make the overall $P_e \leq \eta^{-f(r_n)}$, we need to ensure that $\Pr(\varepsilon_l, l) \leq \eta^{-f(r_n)}$ for $1 \leq l \leq L-1$. However, those conditions are not *essential* in order to achieve the optimal DMT for the parallel channel shown in (6), which only requires the condition (7). Thus stricter code design criteria are required for the rateless channel. One example of such a criterion is the *approximately universal* criterion [3].

Codes being *approximately universal* for parallel channels ensure that the highest error probability when decoding *any* subset of $\{\mathbf{X}_i\}$ in the set of all non-outage events decays *exponentially* in SNR (i.e., in the form of $e^{-\eta^\delta}$ for some $\delta > 0$) under *any* fading distribution, and thus can be ignored compared with the outage probability under the same fading distribution, when the SNR goes to infinity. Specifically, we consider the following parallel MIMO channel which is more general than the one in (6):

$$\mathbf{Y} = \sqrt{\frac{P}{M}} \begin{pmatrix} \mathbf{H}_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{H}_L \end{pmatrix} \begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_L \end{pmatrix} + \begin{pmatrix} \mathbf{N}_1 \\ \vdots \\ \mathbf{N}_L \end{pmatrix} \quad (8)$$

where each channel matrix in $\{\mathbf{H}_i\}$ ($1 \leq i \leq L$) follows an *arbitrary* distribution. In particular, when the matrices in $\{\mathbf{H}_i\}$ are i.i.d. and of the same distributions as the \mathbf{H} in (2), following the same steps as those in [1], it is not difficult to show that the optimal DMT for this system is $d = Lf\left(\frac{r}{L}\right)$ for $0 \leq r \leq L \min(M, N)$. Now, we are ready to state the following theorem considering the performance of rateless codes that are transformed from the approximately universal codes for the parallel channel in (8).

Theorem 2: Suppose a code $[\mathbf{X}_1^T \cdots \mathbf{X}_L^T]^T$ is *approximately universal* for the parallel channel shown in (8) and can achieve the DMT points $(Lr_n, Lf(r_n))$ for $0 \leq r_n \leq \min(M, N)$ when the channel matrices have i.i.d. Rayleigh fading. Then, its transformation $[\mathbf{X}_1 \cdots \mathbf{X}_L]$, when applied to the rateless channel shown in (2) aiming at multiple

multiplexing gains $(r_n, 2r_n, \dots, Lr_n)$, can achieve the DMT shown in *Theorem 1*. ■

Proof: See Appendix B. ■

While approximately universal codes for the general parallel MIMO channel is unknown to the authors, approximately universal codes for parallel SISO channels do exist, and can be transformed directly into good rateless codes for SISO channels. In the following, we apply permutation codes for parallel channels [3] to the rateless channel.

Permutation codes are a class of codes generated from QAM constellations. In the encoding process, a message is mapped into different QAM constellation points across all subchannels. The constellation over one subchannel is a permutation of the points in the constellation over any other subchannel. The permutation is optimized such that the minimal codeword difference is large enough to satisfy the approximate universality criterion. Explicit permutation codes can be constructed using *universally decodable matrices*. We refer the readers to [3] and the references therein for details. It has been shown that permutation codes achieve the optimal DMT for parallel channels and have a particularly simple structure. For example, the codewords are of *unit* length.

Assume the transmission rates over rateless channel are $(R, 2R, \dots, LR)$ bits per channel use. To implement permutation codes, we choose a codebook of size 2^{LR} (messages) for the parallel channel in (8). Each message is mapped into a code $[\mathbf{X}_1^T \cdots \mathbf{X}_L^T]^T$, in which each \mathbf{X}_i is an 2^{LR} -point QAM constellation. The message can be fully recovered as long as any subset of $\{\mathbf{X}_i\}$ can be correctly decoded. Now, we transform this code into the form $[\mathbf{X}_1 \cdots \mathbf{X}_L]$ for the rateless channel. Since $\Pr(\varepsilon_l, l)$ decays exponentially in SNR due to the approximate universality of such codes, the overall error probability is always dominated by that upon receiving all \mathbf{X}_i for *infinitely* high SNR. More precisely, we summarize the above observations as the following corollary.

Corollary 1: Rateless codes that are transformed from permutation codes for parallel channels can offer exactly the same performance as shown in *Theorem 1* over the SISO rateless channel.

Proof: The proof is a direct extension of the proof of *Theorem 2* and is omitted. ■

V. CONCLUSIONS

The performance of rateless codes has been studied for MIMO fading channels in terms of the DMT. The analysis shows that design principles for rateless codes can follow these of the approximately universal codes for parallel MIMO channels. Specifically, it has been shown that for a SISO channel, the formerly developed permutation codes of *unit* length for parallel channels having rate LR can be transformed *directly* into rateless codes of length L having multiple rate levels $(R, 2R, \dots, LR)$, to achieve the desired optimal DMT performance.

APPENDIX

A. Proof of Theorem 1

Define $r_L = Lr_n$. Following the steps in [1], it is easy to show that $p(l) \doteq \eta^{-f(\frac{r_L}{L})}$ for $l \neq 0$. We write the error probability as

$$P_e = \sum_{l=1}^{L-1} (1 - p(l)) \Pr(\varepsilon_l) + \Pr(\varepsilon_L, L). \quad (9)$$

In (9), $\Pr(\varepsilon_l)$ is error probability when $lI_b \geq LTR$, where I_b is the mutual information of the channel in each block. Using Fano's inequality we can obtain the error probability lower bound [1]:

$$P_e \geq \Pr(\varepsilon_L, L) \geq \eta^{-f(\frac{r_L}{L})}.$$

Since $r \leq r_L$, we have $\eta^{-f(\frac{r_L}{L})} \geq \eta^{-f(\frac{r}{L})}$, and thus the desired performance upper bound is obtained.

Now we prove the achievability part. Consider $\Pr(\varepsilon_l)$. Following the same argument as in the proof of Theorem 10.1.1 in [8], we get

$$\Pr(\varepsilon_l) \leq 3\epsilon \quad (10)$$

for sufficiently large T . Note that a very similar argument has been made in *Lemma 1* in [7], although it is claimed there that both T and L are required to be sufficiently large in order to satisfy (10). Now (9) can be further rewritten as

$$\begin{aligned} P_e &\leq 3(L-1)\epsilon + \eta^{-f(\frac{r_L}{L})} + (1 - p(L)) \Pr(\varepsilon_L) \\ &\doteq \eta^{-f(\frac{r_L}{L})}. \end{aligned} \quad (11)$$

Note that

$$\bar{R} \doteq \frac{LR}{1 + \sum_{i=1}^{L-1} \eta^{-f(\frac{r_L}{L})}} \doteq LR$$

for $0 \leq r_L < \min(M, N)$. Thus $r = r_L$ and diversity gain $f(\frac{r}{L})$ is achievable in the range $0 \leq r < \min(M, N)$. Note that $r_L = Lr_n$, and thus we have $d = f(r_n)$ for

$$r = r_n L, 0 \leq r_n < \frac{\min(M, N)}{L}.$$

So far we have only considered the scenario in which $r_n < \frac{\min(M, N)}{L}$. Now the question to ask is what happens if we increase the value of r_n to $\frac{\min(M, N)}{L}$ and beyond. In this scenario, $f(\frac{r_L}{L}) = 0$, and thus $\bar{R} \doteq \frac{LR}{2}$. The message rate r is decreased to $r_L/2$ due to the fact that after the first block the receiver has no chance of decoding the message correctly and it always needs the second block. However, the system error probability P_e is not changed. Therefore the message rate becomes

$$r = r_n \cdot \frac{L}{2}, \frac{\min(M, N)}{L} \leq r_n < \frac{2 \min(M, N)}{L}, \quad (12)$$

and the system DMT becomes

$$d = f\left(\frac{2r}{L}\right), \frac{\min(M, N)}{2} \leq r < \min(M, N). \quad (13)$$

Similarly, when r reaches $\min(M, N)$ again, i.e., r_n reaches $\frac{2 \min(M, N)}{L}$, $f(\frac{r_L}{L}) = f(\frac{2r}{L}) = 0$. Thus $\bar{R} \doteq \frac{LR}{3}$ and

$$r = r_n \cdot \frac{L}{3}, \frac{2 \min(M, N)}{L} \leq r_n < \frac{3 \min(M, N)}{L}; \quad (14)$$

the system DMT becomes

$$d = f\left(\frac{3r}{L}\right), \frac{2 \min(M, N)}{3} \leq r < \min(M, N). \quad (15)$$

Continuing following the above until $\bar{R} \doteq R$, we obtain the desired result and the proof is completed.

B. Proof of Theorem 2

Assume that the system in (6) transmits at a rate $LR = r_L \log_2 \eta$. The probability of any decoding error can be upper bounded by [1]

$$P \leq P_O + P_{e|O^c}$$

where P_O is the outage probability and $P_{e|O^c}$ is the average error probability given that the channel is not in outage. Approximately universality means that for such codes $P_{e|O^c} = e^{-\eta^\delta}$ under any fading distribution. For the system in (8), these include the fading distributions in which $\mathbf{H}_1 = \dots = \mathbf{H}_l$ follow the same distribution as the \mathbf{H} in (2) and $\mathbf{H}_{l+1} = \dots = \mathbf{H}_L \equiv 0$ for all $1 \leq l \leq L-1$. When such codes are transformed into the rateless channels shown in (2), it is a simple matter to show that

$$\Pr(\varepsilon_l) = P_{e|O^c} = e^{-\eta^\delta}$$

for any $1 \leq l \leq L$, where $\Pr(\varepsilon_l)$ is given in (9). Thus the system error probability for the rateless channel in (2) is always upper bounded by

$$P_e \leq L e^{-\eta^\delta} + \eta^{-f(\frac{r_L}{L})} \doteq \eta^{-f(\frac{r_L}{L})}.$$

The rest of the proof follows that of *Theorem 1* and is omitted.

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